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Calculating the prepotential by localization on the moduli space of instantons

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Abstract: We describe a new technique for calculating instanton effects in supersymmetric gauge theories applicable on the Higgs or Coulomb branches. In these situations the instantons are constrained and a potential is generated on the instanton moduli space. Due to existence of a nilpotent fermionic symmetry the resulting integral over the instanton moduli space localizes on the critical points of the potential. Using this technology we calculate the one- and two-instanton contributions to the prepotential of $\mathrm{SU}(N)$ gauge theory with $\mathcal{N} = 2$ supersymmetry and show how the localization approach yields the prediction extracted from the Seiberg-Witten curve. The technique appears to extend to arbitrary instanton number in a tractable way.

Keywords: Nonperturbative Effects, Solitons Monopoles and Instantons.

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1. Introduction

Instantons are particularly fascinating in the context of supersymmetric theories. Certain physical quantities are known on the basis of non-renormalization theorems to receive non-perturbative instanton contributions which are semi-classically exact. For instance in this work we will be concerned with the prepotential of an $\mathcal{N} = 2$ supersymmetric gauge theory with — for definiteness — $\mathrm{SU}(N)$ gauge symmetry. Of course, the celebrated theory of Seiberg and Witten [1] gives an entirely different way to calculate the instanton expansion of the prepotential, however, this does not detract from the dream of calculating the prepotential directly from the path integral using semi-classical techniques. For one thing consistency shows that the Euclidean path integral does actually have something to do with Minkowski space field theory beyond perturbation theory and also that the remarkable edifice of Seiberg-Witten theory is consistent with more conventional approaches.

The problem is that instanton calculations are not easy. In the semi-classical limit the functional integral reduces to a sum over terms involving integrals over the moduli spaces of instantons $\mathfrak{M}_{k,N}$ of charge k.[1] The $4kN$-dimensional moduli space $\mathfrak{M}_{k,N}$ can be constructed via the Atiyah-Drinfeld-Hitchin-Manin (ADHM) construction [2]; however, this realizes $\mathfrak{M}_{k,N}$ as a hyper-Kähler quotient and so the construction is only implicit: there are complicated non-linear constraints which have not at present been solved, for general N, beyond $k = 1$ (except on generic orbits for $N \geq 2k$ [3, 4]). Even in the case of a single instanton, where the "ADHM constraints" can be solved, the resulting integral over $\mathfrak{M}_{1,N}$ which give the one-instanton contribution to the prepotential is rather complicated for

[1]Here, N keeps track of the gauge group.

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$N > 2$, as one can appreciate by following the machinations of ref. [5].[2] A direct attack on the two-instanton contribution can only be performed for the gauge group SU(2) (actually realized via the Sp(1) ADHM construction) [6, 7]. The two-instanton calculation was a *tour de force* in its own right and sadly the extension to SU(N) and/or $k > 2$ looks a touch unrealistic. A new idea is urgently needed and this is what the present work provides.

The prepotential of an $\mathcal{N} = 2$ theory with N_F fundamental hypermultiplets has the form[3]

$$\mathcal{F} = \mathcal{F}_{\text{classical}} + \mathcal{F}_{\text{one-loop}} + \frac{1}{2\pi i}\sum_{k=1}^{\infty} \mathcal{F}_k \Lambda^{k(2N-N_F)} . \tag{1.1}$$

where Λ is the usual scale of the Pauli-Villars regularization scheme.[4] The coefficient $\mathcal{F}_k$ is given by

$$\mathcal{F}_k = \widehat{\mathcal{Z}}^{(\mathcal{N}=2,N_F)}_{k,N} , \tag{1.2}$$

the "centred instanton partition function" defined as an integral over the suitably $\mathcal{N} = 2$ supersymmetrized version of the centred k-instanton moduli space $\widehat{\mathfrak{M}}_{k,N}$.[5] If $\boldsymbol{\omega}^{(\mathcal{N}=2,N_F)}$ is the $\mathcal{N} = 2$ supersymmetric volume form, then

$$\widehat{\mathcal{Z}}^{(\mathcal{N}=2,N_F)}_{k,N} = \int_{\widehat{\mathfrak{M}}_{k,N}} \boldsymbol{\omega}^{(\mathcal{N}=2,N_F)}\, e^{-S} . \tag{1.3}$$

The quantity S is the instanton effective action which depends on the VEVs parameterizing the Coulomb branch in addition to the masses of hypermultiplets. This action is a direct manifestation of the fact that instantons are not exact solutions of the equation-of-motion on the Coulomb branch, rather they should be treated as constrained instantons *à la* Affleck [8, 9] (see also the in-depth discussion in refs. [4, 6]).

The partition function (1.3) can be obtained as the dimensional reduction to zero dimensions of the partition function of a two-dimensional $\mathcal{N} = (0, 4)$ supersymmetric σ-model with $\widehat{\mathfrak{M}}_{k,N}$ as target [4]. It is possible to linearize this σ-model by introducing a non-dynamical U(k) vector multiplet containing a two-dimensional gauge field. Integrating out the vector multiplet simply implements the hyper-Kähler quotient construction of $\mathfrak{M}_{k,N}$. The VEVs of the four-dimensional gauge theory can then be incorporated via a non-trivial dimensional reduction. It is important to bear in mind that the original SU(N) gauge symmetry of the four-dimensional $\mathcal{N} = 2$ theory is realized as a global symmetry of this zero-dimensional "field", or matrix, theory. Will find the linearized formulation of the partition function particularly advantageous for our calculations.

The key point, following refs. [10, 11],[6] is that there exists a fermionic symmetry $\mathcal{Q}$ defined as a particular combination of the supersymmetries for which the instanton effective

[2] In fact, if one inspects the approach of ref. [5] closely one will see that the ADHM constraints are never actually solved: they are replaced by integrals over Lagrange multipliers, a trick which we will use below.

[3] In the case of the finite theory, obtained when $N_F = 2N$, one must replace the factor Λ^{2N-N_F} by $e^{2\pi i\tau}$ where, as usual, $\tau = 4\pi i/g^2 + \theta/2\pi$.

[4] For $N_F = 2N$ the factor Λ^{2N-N_F} must by replaced by $e^{2\pi i\tau}$.

[5] The centred moduli space has the overall position of the instanton configuration factored off: $\mathfrak{M}_{k,N} = \mathbb{R}^4 \times \widehat{\mathfrak{M}}_{k,N}$.

[6] The former reference considers the analogous construction in the context of the $\mathcal{N} = 4$ theory.

action has the form

$$S = \mathcal{Q}\,\Xi + \Gamma\,, \tag{1.4}$$

where $\mathcal{Q}\Gamma = 0$. The operator $\mathcal{Q}$ can be shown to be nilpotent on quantities invariant under the $\mathrm{U}(k)$ auxiliary symmetry group and also under the $\mathrm{U}(1)^{N-1} \subset \mathrm{SU}(N)$ subgroup of the global symmetries picked out by the VEVs of the four-dimensional $\mathcal{N}=2$ theory. In particular, the action S and integration measure $\boldsymbol{\omega}^{(\mathcal{N}=2,N_F)}$ are $\mathcal{Q}$-invariant. Now consider the more general integral

$$\widehat{\mathfrak{Z}}^{(\mathcal{N}=2,N_F)}_{k,N}(s) = \int_{\widehat{\mathfrak{M}}_{k,N}} \boldsymbol{\omega}^{(\mathcal{N}=2,N_F)}\, \exp\big(-s^{-1}\mathcal{Q}\,\Xi - \Gamma\big)\,. \tag{1.5}$$

We then have

$$\frac{\partial \widehat{\mathfrak{Z}}^{(\mathcal{N}=2,N_F)}_{k,N}(s)}{\partial s} = s^{-2}\int_{\widehat{\mathfrak{M}}_{k}} \boldsymbol{\omega}^{(\mathcal{N}=2,N_F)}\, \mathcal{Q}\Big\{\Xi\,\exp\big(-s^{-1}\mathcal{Q}\,\Xi - \Gamma\big)\Big\}\,, \tag{1.6}$$

using the fact that $\mathcal{Q}^2\,\Xi = \mathcal{Q}\Gamma = 0$. Since the volume form is $\mathcal{Q}$-invariant the right-hand side of (1.6) vanishes. Consequently, $\widehat{\mathfrak{Z}}^{(\mathcal{N}=2,N_F)}_{k,N}(s)$ is independent of s and, therefore, it can be evaluated in the limit $s \to 0$ where the integral is dominated by the critical points of $\mathcal{Q}\,\Xi$. Since the result is independent of s, under favourable circumstances — which will be shown to hold in the present application — the gaussian approximation is exact (for references to this kind of localization in the physics literature see refs. [12]–[15] and references therein). In the present case, the presence of an effective action for instantons arises from the fact that instanton are constrained on the Coulomb branch. The action acts as a potential on $\mathfrak{M}_{k,N}$ which penalizes large instantons and the critical points can be identified with the configurations where all the instantons have shrunk to zero size. The potential problem is that the theory of localization is most easy to apply to situations involving compact spaces without boundary. In the case at hand, the instanton moduli space is obviously not compact and, in addition, has conical singularities precisely when instantons shrink to zero size. On top of this, the critical-point sets are non-compact and have their own singularities when the point-like instantons can becomes arbitrarily separated in $\mathbb{R}^4$ or they can come together at the same point in $\mathbb{R}^4$, respectively. We will argue that the potential problems are removed by considering the partition function in the $\mathcal{N}=2$ theory defined on a *non-commutative* spacetime.[7] In such a theory, instantons are described by a deformed moduli space $\mathfrak{M}^{(\zeta)}_{k,N}$, where ζ^c, $c = 1, 2, 3$, are non-commutativity parameters, that is a smooth resolution of $\mathfrak{M}_{k,N}$: instantons can no longer shrink to zero size on a non-commutative spacetime.[8] Of course the question then arises as to whether it is legitimate in the context of the prepotential to replace the instanton moduli space by its smooth resolution? It has been argued in ref. [24] that this is indeed the case and that the physical content of the prepotential of the non-commutative theory is the same as that of the commutative theory.[9]

[7] Actually in order to define a non-commutative version of the theory we have to take a gauge group $\mathrm{U}(N)$. The addition of the extra $\mathrm{U}(1)$ factor does not affect the commutative theory since it trivially decouples.

[8] Various aspects of instantons in non-commutative theories are considered in refs. [16, 17]–[23].

[9] There are differences which turn out to be non-physical, as we shall see.

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Before proceeding we should point out that the construction of a BRST-type operator $\mathcal{Q}$ in the context of the $\mathcal{N}=2$ instanton calculus was first proposed in refs. [11, 25, 26]. (In particular, the latter reference is most closely related to the approach that we adopt in this paper.) However, these references emphasized the relation with the topological version of the original gauge theory. However, these references did not go on to use the existence of $\mathcal{Q}$ to develop a calculational technique based on localization. A nilpotent fermionic symmetry was also constructed in the context of the $\mathcal{N}=4$ instanton calculus in ref. [10] where localization was first proposed as a method to calculate, in this case the $\mathcal{N}=4$, instanton partition function. Some recent papers [27] have also considered the $\mathcal{Q}$-operator and the $\mathcal{N}=2$ instanton calculus, although they use the equivalent language of differential forms in which $\mathcal{Q}$ corresponds to an equivariant exterior derivative on the instanton moduli space. These references then go some way towards interpreting $\mathcal{F}_k$ as a topological intersection number.

2. The instanton calculus and localization

In this section, we will briefly review relevant aspects of the instanton calculus, construct the fermionic symmetry and prove the properties of the centred instanton partition function that we described above. For a more detailed discussion of the instanton calculus we refer the reader to refs. [3, 4, 5].

The ADHM construction of instantons involves a set of over-complete collective coordinates $\{w_{\dot\alpha}, a'_n\}$. Here, $w_{\dot\alpha}$, $\dot\alpha=1,2$, are $N\times k$ matrices with elements $w_{ui\dot\alpha}$ and a'_n, $n=1,2,3,4$, are $k\times k$ hermitean matrices.[10] The instanton moduli space $\mathfrak{M}_{k,N}$ is then obtained as a hyper-Kähler quotient [28] by the group $\mathrm{U}(k)$ acting on the variables as

$$w_{\dot\alpha}\to w_{\dot\alpha}\,\mathcal{U}\,,\qquad a'_n\to\mathcal{U}^\dagger\,a'_n\,\mathcal{U}\,,\tag{2.1}$$

$\mathcal{U}\in\mathrm{U}(k)$. The hyper-Kähler quotient involves a two-stage process. First one defines the "level set" $\mathfrak{N}$, to be the subspace of the "mother" space, in this case simply $\mathbb{R}^{4k(N+k)}$ parameterized by $\{w_{\dot\alpha}, a'_n\}$, on which the $3k^2$ following "ADHM constraints" are satisfied (known also as the vanishing of the moment maps):[11]

$$\tau^{c\dot\alpha}{}_{\dot\beta}\big(\bar w^{\dot\beta}w_{\dot\alpha}+\bar a'^{\dot\beta\alpha}a'_{\beta\dot\alpha}\big)=\zeta^c 1_{[k]\times[k]}\,.\tag{2.2}$$

Here, ζ^c are real constants which are set to zero if we wish to describe $\mathfrak{M}_{k,N}$, whereas they are left non-trivial, $\zeta\cdot\zeta>0$, in order to describe the smooth resolution $\mathfrak{M}^{(\zeta)}_{k,N}$. The instanton moduli space is then the ordinary quotient of $\mathfrak{N}$ by the $\mathrm{U}(k)$ action (2.1) which fixes $\mathfrak{N}\subset\mathbb{R}^{4k(N+k)}$. In order to define the centred moduli space $\widehat{\mathfrak{M}}^{(\zeta)}_{k,N}$ one simply imposes the fact that a'_n are traceless. The relation of the instanton calculus to the hyper-Kähler quotient construction has been emphasized in refs. [4, 30].

[10] $\mathrm{SU}(N)$ gauge indices are denoted $u,v=1,\dots,N$ and "instanton" indices are denoted $i,j=1,\dots,k$, where k is the instanton charge.

[11] Here, τ^c, $c=1,2,3$, are the Pauli matrices, $a'_{\alpha\dot\alpha}=a'_n\sigma_{\alpha\dot\alpha n}$ and $\bar a'^{\dot\alpha\alpha}=a'_n\bar\sigma_n^{\dot\alpha\alpha}$, where $\sigma_n=(i\tau^c,1_{[2]\times[2]})$ and $\bar\sigma_n=(-i\tau^c,1_{[2]\times[2]})$. Spinor indices $\dot\alpha,\dot\beta,\dots$ are raised and lowered with the ϵ-tensor as in [29]. Finally, $\bar w_{\dot\alpha}\equiv(w_{\dot\alpha})^\dagger$.

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We remark at this point that the construction of $\mathfrak{M}_{k,N}$ is valid even in the case $N = 1$. In this case when $\zeta^c = 0$ it is easy to see that the constraints (2.2) are solved with $w_{\dot\alpha} = 0$ and a'_n diagonal; hence, taking into account that the diagonal form for $a' - N$ is fixed by permutations, we have

$$\mathfrak{M}_{k,1} = \frac{\mathbb{R}^4 \times \cdots \times \mathbb{R}^4}{S_k}\,, \tag{2.3}$$

where the symmetric group S_k acts by permutation. One can show that the resulting expression for the U(1) gauge field is pure gauge. In the non-commutative U(1) theory, instantons become non-trivial and $\mathfrak{M}^{(\zeta)}_{k,1}$ is a smooth resolution of the singular space (2.3) [16, 31]. For example, the centred moduli space $\widehat{\mathfrak{M}}^{(\zeta)}_{2,1}$ is a smooth four-dimensional hyper-Kähler space which can be shown to be the Eguchi-Hanson manifold [18].

In an $\mathcal{N} = 2$ supersymmetric theory an instanton has a set of Grassmann collective coordinates which parameterize the $4kN$ zero modes of the Dirac operator in the instanton background. In the supersymmetric extension of the ADHM construction we first define a set of over-complete Grassmann variables $\{\mu^A, \bar\mu^A, \mathcal{M}'^A_\alpha\}$. Here $A = 1, 2$ is an index of the SU(2) R-symmetry,[12] μ^A is an $N \times k$ matrix with elements μ^A_{ui}, $\bar\mu^A$ is a $k \times N$ matrix with elements $\bar\mu^A_{iu}$ and $\mathcal{M}'^A_\alpha$ are $k \times k$ matrices with elements $(\mathcal{M}'^A_\alpha)_{ij}$. This over-complete set of variables is subject to fermionic analogues of the ADHM constraints:

$$\bar\mu^A w_{\dot\alpha} + \bar w_{\dot\alpha}\mu^A + [\mathcal{M}'^{\alpha A}, a'_{\alpha\dot\alpha}] = 0\,. \tag{2.4}$$

For the centred moduli space we also impose the tracelessness of the matrices $\mathcal{M}'^A_\alpha$. Geometrically we can identify the independent Grassmann collective coordinates with tangent vectors to $\mathfrak{M}^{(\zeta)}_{k,N}$.

When we include N_F fundamental hypermultiplets there are additional Grassmann collective coordinates $\mathcal{K}$ and $\tilde{\mathcal{K}}$ which are $k \times N_F$ and $N_F \times k$ matrices with elements $\mathcal{K}_{if}$ and $\tilde{\mathcal{K}}_{fi}$, $f = 1, \ldots, N_F$, respectively. These coordinates are not subject to any constraints.

We now specify the volume form $\boldsymbol{\omega}^{(\mathcal{N}=2,N_F)}$. For the moment, we consider the full moduli space $\mathfrak{M}^{(\zeta)}_{k,N}$ rather than its centred version. The integral can be split into a c-number piece times various Grassmann integrals:

$$\int_{\mathfrak{M}^{(\zeta)}_{k,N}} \boldsymbol{\omega}^{(\mathcal{N}=2,N_F)} \equiv \int_{\mathfrak{M}^{(\zeta)}_{k,N}} \boldsymbol{\omega} \times \text{Grassmann integrals}\,. \tag{2.5}$$

The c-number volume form $\boldsymbol{\omega}$ is the canonical volume form on $\mathfrak{M}^{(\zeta)}_{k,N}$. This can be deduced from the hyper-Kähler quotient. The quotient starts from flat space with metric

$$ds^2 = 8\pi^2\, \mathrm{tr}_k\, \left[d\bar w^{\dot\alpha}\, dw_{\dot\alpha} + da'_n\, da'_n\right]. \tag{2.6}$$

A natural metric is then induced on the hyper-Kähler $\mathfrak{M}_{k,N}$ by restriction to $\mathfrak{N}$ and then via a conventional quotient by U(k). If this metric is $g = g_{\mu\nu} dX^\mu\, dX^\nu$, in terms of a set of intrinsic coordinates $\{X^\mu\}$ on $\mathfrak{M}_{k,N}$, then

$$\int_{\mathfrak{M}^{(\zeta)}_{k,N}} \boldsymbol{\omega} = \int \sqrt{\det g}\, \prod_\mu \frac{dX^\mu}{\sqrt{2\pi}}\,. \tag{2.7}$$

[12]These indices are raised and lowered with the usual ϵ-tensor of ref. [29].

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Unfortunately, in general beyond $k=1$, the ADHM constraints have not been solved and so it is not possible to find an explicit set of intrinsic coordinates on $\mathfrak{M}^{(\zeta)}_{k,N}$. However, we can write down an implicit expression which leaves the ADHM constraints explicit and where we don't fix the $\mathrm{U}(k)$ symmetry [32, 33]. One of the main conclusions from our analysis is that we can make progress even when we *cannot* solve the ADHM constraints. This implicit form for the collective coordinate integration measure for the full moduli space, following refs. [5, 3] for the $\mathrm{SU}(N)$ case, is[13]

$$\int_{\mathfrak{M}^{(\zeta)}_{k,N}}\boldsymbol{\omega}=\frac{2^{-k(k-1)/2}(2\pi)^{2kN}}{\mathrm{Vol}\,\mathrm{U}(k)}\int d^{2kN}w\,d^{2kN}\bar w\,d^{4k^2}a'\big|\det{}_{k^2}\boldsymbol{L}\big|\times \times\prod_{r=1}^{k^2}\prod_{c=1}^{3}\delta\Big(\frac12\mathrm{tr}_kT^r\big(\tau^{c\dot\alpha}{}_{\dot\beta}(\bar w^{\dot\beta}w_{\dot\alpha}+\bar a'^{\dot\beta\alpha}a'_{\alpha\dot\alpha})-\zeta^c1_{[k]\times[k]}\big)\Big)\,. \tag{2.8}$$

Here, $\boldsymbol{L}$ is an operator on $k\times k$ matrices defined by

$$\boldsymbol{L}\cdot\Omega=\frac12\{\bar w^{\dot\alpha}w_{\dot\alpha},\Omega\}+[a'_n,[a'_n,\Omega]]\,. \tag{2.9}$$

The fact that (2.8) follows from the hyper-Kähler quotient construction was shown in ref. [30] (see also the review [4]). Geometrically, the matrix element $\mathrm{tr}_k(T^r\boldsymbol{L}T^s)$, for two generators T^r,T^s of $\mathrm{U}(k)$, gives the inner-product of the corresponding Killing vectors on the mother space of the hyper-Kähler quotient [4].

There are two Grassmann pieces to the collective coordinate integral (2.5). The first corresponds to the zero modes of the adjoint-valued fermions which involve integrals over $\{\mu^A,\bar\mu^A,\mathcal{M}'^A_\alpha\}$ subject to the fermionic analogues of the ADHM constraints (2.4). The form of the integral is dictated by the hyper-Kähler quotient construction [4]. For $\mathcal{N}$ supersymmetries these integrals are [5, 33, 3][14]

$$\prod_{A=1}^{\mathcal{N}}\Big\{2^{-kN}\pi^{-2kN}\int d^{kN}\mu^A\,d^{kN}\bar\mu^A\,d^{2k^2}\mathcal{M}'^A\,\big|\det{}_{k^2}\,\boldsymbol{L}\big|^{-1}\times \times\prod_{r=1}^{k^2}\prod_{\dot\alpha=1}^{2}\delta\Big(\mathrm{tr}_k\,T^r(\bar\mu^Aw_{\dot\alpha}+\bar w_{\dot\alpha}\mu^A+[\mathcal{M}'^{\alpha A},a'_{\alpha\dot\alpha}])\Big)\Big\}\,. \tag{2.10}$$

To complete the collective coordinate integral there are integrals over the Grassmann collective coordinates arising from the fundamental hypermultiplets:

$$\pi^{-2kN_F}\int d^{kN_F}\mathcal{K}\,d^{kN_F}\tilde{\mathcal{K}}\,. \tag{2.11}$$

To define the centred partition function we have to separate out the integrals over the trace parts of a'_n and $\mathcal{M}'^A_\alpha$. These collective coordinates are generated by translations and global supersymmetry transformations on the instanton configuration. Taking into account

[13]In the following we use the following definitions. The integrals over a'_n and the arguments of the δ-functions are defined with respect to the generators of $\mathrm{U}(k)$ in the fundamental representation, normalized so that $\mathrm{tr}_k\,T^rT^s=\delta^{rs}$. The volume of the $\mathrm{U}(k)$ is the constant $2^k\pi^{k(k+1)/2}/\prod_{i=1}^{k-1}i!$.

[14]Unlike [29] out convention for integrating a two-component spinor ψ_α is $\int d^2\psi\equiv\int d\psi_1\,d\psi_2$.

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the normalization of the associated zero modes the prescription for defining the centred partition function is

$$\int_{\mathfrak{M}^{(\zeta)}_{k,N}} \boldsymbol{\omega}^{(\mathcal{N}=2,N_F)} = \frac{4}{\pi^2}\int d^4(\mathrm{tr}_k\, a')\, d^4(\mathrm{tr}_k\, \mathcal{M}') \times \int_{\widehat{\mathfrak{M}}^{(\zeta)}_{k,N}} \boldsymbol{\omega}^{(\mathcal{N}=2,N_F)} . \tag{2.12}$$

The final ingredient of the centred partition function is the instanton effective action. This has been determined in [5] (see the review [4] for full details). The expression depends on the VEVs of the scalar fields. Sometimes it will be convenient to think of the VEVs as being a real 2-vector $\boldsymbol{\phi}$ and sometimes as the (anti-)holomorphic quantity ϕ ($\phi^\dagger$) with

$$\boldsymbol{\phi} = (\mathrm{Re}\,\phi, \mathrm{Im}\,\phi) . \tag{2.13}$$

The quantity ϕ is then a diagonal $N\times N$ matrix with elements ϕ_u. The traceless condition implies $\sum_{u=1}^N \phi_u = 0$. The expression for the effective action is

$$\begin{aligned} S = 4\pi^2\, \mathrm{tr}_k \Bigg\{ & -\frac{i}{2}\bar\mu^A\phi^\dagger\mu_A + \bar w^{\dot\alpha}\boldsymbol{\phi}^2 w_{\dot\alpha} + \qquad (2.14)\\ & + \left(\frac14\sum_{f=1}^{N_F}\mathcal{K}_f\tilde{\mathcal{K}}_f - \bar w^{\dot\alpha}\phi^\dagger w_{\dot\alpha}\right)\boldsymbol{L}^{-1}\left(-\frac{i}{2}\bar\mu^A\mu_A - \frac{i}{2}\mathcal{M}'^{\alpha A}\mathcal{M}'_{\alpha A} + \bar w^{\dot\alpha}\phi w_{\dot\alpha}\right)\Bigg\} . \end{aligned}$$

Note that the action penalizes large instantons as is characteristic of constrained instantons.

The centred instanton partition function can be formulated in an equivalent but more useful way by introducing auxiliary variables in the form of: $\boldsymbol{\chi} = (\chi_1, \chi_2)$ a 2-vector of hermitean $k\times k$ matrices; D^c, $c = 1,2,3$, three hermitean $k\times k$ matrices; and $k\times k$ matrices of Grassmann superpartners $\bar\psi^{\dot\alpha}_A$. Using these variables, the partition function can be written in a completely "linearized" form:

$$\begin{aligned} \widehat{\mathcal{Z}}^{(\mathcal{N}=2,N_F)}_{k,N} = & \frac{2^{-k(k-1)/2-2}\pi^{-4k^2-2kN-2kN_F+2}}{\mathrm{Vol}\,\mathrm{U}(k)}\int d^{2kN}w\, d^{2kN}\bar w\, d^{4(k^2-1)}a'\, d^{3k^2}D\, d^{2k^2}\boldsymbol{\chi}\, \times \\ & \times d^{2kN}\mu\, d^{2kN}\bar\mu\, d^{4(k^2-1)}\mathcal{M}'\, d^{4k^2}\bar\psi\, d^{kN_F}\mathcal{K}\, d^{kN_F}\tilde{\mathcal{K}}\, \exp(-S) , \end{aligned} \tag{2.15}$$

where the action is

$$\begin{aligned} S = 4\pi^2\, \mathrm{tr}_k \Bigg\{ & \big|w_{\dot\alpha}\boldsymbol{\chi} + \boldsymbol{\phi} w_{\dot\alpha}\big|^2 - [\boldsymbol{\chi}, a'_n]^2 + \frac{i}{2}\bar\mu^A(\mu_A\chi^\dagger + \phi^\dagger\mu_A) + \\ & + \frac{i}{4}\mathcal{M}'^{\alpha A}[\mathcal{M}'_{\alpha A}, \chi^\dagger] + \frac14\sum_{f=1}^{N_F}(m_f - \chi)\mathcal{K}_f\tilde{\mathcal{K}}_f\Bigg\} + S_{\mathrm{L.m.}} . \end{aligned} \tag{2.16}$$

Here, $\chi = \chi_1 + i\chi_2$. Notice that the $k\times k$ matrices D^c and $\bar\psi^{\dot\alpha}_A$ act as Lagrange multipliers for the bosonic and fermionic ADHM constraints (2.2) and (2.4) through the final term in the action

$$S_{\mathrm{L.m.}} = -4i\pi^2\mathrm{tr}_k\Big\{\bar\psi^{\dot\alpha}_A\big(\bar\mu^A w_{\dot\alpha} + \bar w_{\dot\alpha}\mu^A + [\mathcal{M}'^{\alpha A}, a'_{\alpha\dot\alpha}]\big) + D^c\big(\tau^{c\dot\alpha}{}_{\dot\beta}(\bar w^{\dot\beta}w_{\dot\alpha} + \bar a'^{\dot\beta\alpha}a'_{\alpha\dot\alpha}) - \zeta^c\big)\Big\} . \tag{2.17}$$

The previous form of the collective coordinate integral, obtained by concatenating (2.8), (2.10) and (2.11), is recovered by integrating out the auxiliary variables $\{\boldsymbol{\chi}, D^c, \bar\psi^{\dot\alpha}_A\}$.

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The instanton effective action (2.16) (or (2.14)) is invariant under four supersymmetries corresponding to precisely half the number of the $\mathcal{N}=2$ theory. On the linearized system, the transformations can be written as

$$\delta a'_{\alpha\dot\alpha} = i\bar\xi_{\dot\alpha A}\mathcal{M}'^A_\alpha\,, \qquad \delta\mathcal{M}'^A_\alpha = 2\bar\xi^{\dot\alpha A}[a'_{\alpha\dot\alpha},\chi]\,, \tag{2.18a}$$
$$\delta w_{\dot\alpha} = i\bar\xi_{\dot\alpha A}\mu^A\,, \qquad \delta\mu^A = 2\bar\xi^{\dot\alpha A}\big(w_{\dot\alpha}\chi+\phi w_{\dot\alpha}\big)\,, \tag{2.18b}$$
$$\delta\chi = 0\,, \qquad \delta\chi^\dagger = 2i\bar\xi^A_{\dot\alpha}\bar\psi^{\dot\alpha}_A\,, \tag{2.18c}$$
$$\delta\bar\psi^{\dot\alpha}_A = \frac{1}{2}[\chi^\dagger,\chi]\bar\xi^{\dot\alpha}_A - iD^c\tau^{c\dot\alpha}{}_{\dot\beta}\bar\xi^{\dot\beta}_A\,. \qquad \delta\mathcal{K}=\delta\tilde{\mathcal{K}}=0\,. \tag{2.18d}$$

One can interpret the partition function in the linearized form (2.15) as the dimensional reduction of a two-dimensional $\mathcal{N}=(0,4)$ supersymmetric gauged linear σ-model [4]. In this interpretation χ is the U(k) two-dimensional gauge field forming a vector multiplet of supersymmetry along with $\bar\psi^{\dot\alpha}_A$ and D^c. These variables have no kinetic term (in two dimensions) and on integrating them out one recovers a non-linear σ-model with the hyper-Kähler space $\widehat{\mathfrak{M}}^{(\zeta)}_{k,N}$ as target.

From the supersymmetry transformations we can define corresponding supercharges via $\delta=\xi_{\dot\alpha A}Q^{\dot\alpha A}$. The fermionic symmetry we are after can then be defined as

$$\mathcal{Q} = \epsilon_{\dot\alpha A}Q^{\dot\alpha A}\,. \tag{2.19}$$

Notice that the definition of $\mathcal{Q}$ mixes up spacetime and R-symmetry indices as is characteristic of topological twisting. The action of $\mathcal{Q}$ on the variables is[15]

$$\mathcal{Q}\,w_{\dot\alpha} = i\epsilon_{\dot\alpha A}\mu^A\,, \qquad \mathcal{Q}\,\mu^A = -2\epsilon^{\dot\alpha A}(w_{\dot\alpha}\chi+\phi w_{\dot\alpha})\,, \tag{2.20a}$$
$$\mathcal{Q}\,a'_{\alpha\dot\alpha} = i\epsilon_{\dot\alpha A}\mathcal{M}'^A_\alpha\,, \qquad \mathcal{Q}\,\mathcal{M}'^A_\alpha = -2\epsilon^{\dot\alpha A}[a'_{\alpha\dot\alpha},\chi]\,, \tag{2.20b}$$
$$\mathcal{Q}\,\chi = 0\,, \qquad \mathcal{Q}\,\chi^\dagger = 2i\delta^A{}_{\dot\alpha}\bar\psi^{\dot\alpha}_A\,, \tag{2.20c}$$
$$\mathcal{Q}\,\bar\psi^{\dot\alpha}_A = \frac{1}{2}\delta^{\dot\alpha}{}_A[\chi^\dagger,\chi] - iD^c\tau^{c\dot\alpha}{}_{\dot\beta}\delta^{\dot\beta}{}_A\,, \qquad \mathcal{Q}\,\mathcal{K}=\mathcal{Q}\,\tilde{\mathcal{K}}=0\,. \tag{2.20d}$$

It is straightforward to show that $\mathcal{Q}$ is nilpotent up to an infinitesimal U$(k)\times$ SU(N) transformation generated by χ and ϕ. For example,

$$\mathcal{Q}^2\,w_{\dot\alpha} = 2i(w_{\dot\alpha}\chi+\phi w_{\dot\alpha})\,. \tag{2.21}$$

It can then be shown that the instanton effective action assumes the form (1.4) with

$$\Xi = 4\pi^2\,\mathrm{tr}_k\Big\{\frac{1}{2}\epsilon_{\dot\alpha A}\bar w^{\dot\alpha}\big(\mu^A\chi^\dagger+\phi^\dagger\mu^A\big)+\frac{1}{4}\epsilon_{\dot\alpha A}\bar a'^{\dot\alpha\alpha}[\mathcal{M}'^A_\alpha,\chi^\dagger]+ \\ +\delta^A{}_{\dot\alpha}\bar\psi^{\dot\beta}_A\left(\bar w^{\dot\alpha}w_{\dot\beta}+\bar a'^{\dot\alpha\alpha}a'_{\alpha\dot\beta}-\frac{1}{2}\tau^{c\dot\alpha}{}_{\dot\beta}\zeta^c\right)\Big\} \tag{2.22}$$

and

$$\Gamma = \pi^2\sum_{f=1}^{N_F}\mathrm{tr}_k\big((m_f-\chi)\mathcal{K}_f\tilde{\mathcal{K}}_f\big)\,. \tag{2.23}$$

[15]This action is essentially identical to the one defined in ref. [11] after integrating out some additional auxiliary variables. It can also be derived, as explained in ref. [11], from the $\mathcal{Q}$-action defined in the $\mathcal{N}=4$ theory in ref. [10].

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Parenthetically we note that the derivative of the action with respect to the variables ζ^c is $\mathcal{Q}$-exact. This means that the partition function cannot depend smoothly on the non-commutativity parameters ζ^c. Of course, there will be a discontinuity at $\zeta^c = 0$ when singularities appear on $\mathfrak{M}^{(\zeta)}_{k,N}$.

Following the logic of localization we should investigate the critical points of $\mathcal{Q}\,\Xi$. The terms to minimize are, from (2.16),

$$\left|w_{\dot\alpha}\boldsymbol{\chi} + \boldsymbol{\phi} w_{\dot\alpha}\right|^2 - [\boldsymbol{\chi}, a'_n]^2 \,. \tag{2.24}$$

Notice that this is positive semi-definite and the critical points are simply the zeros:

$$w_{\dot\alpha}\boldsymbol{\chi} + \boldsymbol{\phi} w_{\dot\alpha} = [\boldsymbol{\chi}, a'_n] = 0 \,. \tag{2.25}$$

Up to the U(k) auxiliary symmetry, each critical-point set is associated to the partition

$$k \to k_1 + k_2 + \cdots + k_N \,. \tag{2.26}$$

Each $i \in \{1, 2, \ldots, k\}$ is then associated to a given u by a map u_i as follows:

$$\Big\{\underbrace{1, 2, \ldots, k_1}_{u=1}, \underbrace{k_1 + 1, \ldots, k_1 + k_2}_{u=2}, \ldots, \tag{2.27}$$
$$\ldots, \underbrace{k_1 + \cdots + k_{u-1} + 1, \ldots, k_1 + \cdots + k_u}_{u}, \ldots, \ldots, \underbrace{k_1 + \cdots + k_{N-1} + 1, \ldots, k}_{u=N}\Big\}$$

For a given partition the variables have a block diagonal-form

$$\boldsymbol{\chi}_{ij} = -\boldsymbol{\phi}_{u_i}\delta_{ij} \,, \qquad w_{ui\dot\alpha} \propto \delta_{uu_i} \,, \qquad (a'_n)_{ij} \propto \delta_{u_iu_j} \,. \tag{2.28}$$

The critical-point sets have a very suggestive form. Imposing the ADHM constraints (2.2) implies that in the $u^{\rm th}$ block the constraints are those of k_u instantons in a non-commutative U(1) gauge theory. The critical-point set associated to $\{k_1, \ldots, k_N\}$ is then simply

$$\mathfrak{F} = \frac{\mathfrak{M}^{(\zeta)}_{k_1,1} \times \cdots \times \mathfrak{M}^{(\zeta)}_{k_N,1}}{\mathbb{R}^4} \,, \tag{2.29}$$

where the quotient is by the overall centre of the instanton. The factors $\mathfrak{M}^{(\zeta)}_{k,1}$ are the non-commutative U(1) instanton moduli spaces described by the hyper-Kähler quotient construction with $N = 1$. Geometrically, as described in ref. [4], the critical-point sets are precisely the fixed submanifolds of the $\mathrm{U}(1)^{N-1}$ tri-holomorphic Killing vectors on $\widehat{\mathfrak{M}}^{(\zeta)}_{k,N}$ defined by the VEVs.

3. The centred one-instanton partition function

We now use the localization technique to evaluate the centred one-instanton partition function.

The instanton effective action has N critical points, labelled by $v \in \{1, 2, \ldots, N\}$, at which (2.28)

$$\boldsymbol{\chi} = -\boldsymbol{\phi}_v \,, \qquad w_{u\dot\alpha} \propto \delta_{uv} \,. \tag{3.1}$$

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Note that $a'_n = 0$ in the one-instanton sector. Without-loss-of-generality, we choose our non-commutativity parameters

$$\zeta^1 = \zeta^2 = 0\,, \qquad \zeta^3 \equiv \zeta > 0\,. \tag{3.2}$$

In this case the ADHM constraints (2.2) are solved with

$$w_{u\dot\alpha} = \sqrt{\zeta} e^{i\theta} \delta_{uv} \delta_{\dot\alpha 1}\,, \tag{3.3}$$

for an arbitrary phase angle θ. The integrals over $w_{v\dot\alpha}$ are then partially saturated by the three δ-functions in (2.8) that impose the ADHM constraints, leaving a trivial integral over the phase angle θ:

$$\int d^2 w_v\, d^2 \bar w_v \prod_{c=1}^{3} \delta\left(\frac{1}{2}\tau^{c\dot\alpha}{}_{\dot\beta}\big(\bar w_v^{\dot\beta} w_{v\dot\alpha} - \zeta\delta^{c3}\big)\right) = 8\pi\zeta^{-1}\,. \tag{3.4}$$

The δ-functions for the Grassmann ADHM constraints in (2.10) are saturated by the integrals over $\{\mu_v^A, \bar\mu_v^A\}$:

$$\int d\mu_v^A\, d\bar\mu_v^A \prod_{\dot\alpha=1}^{2} \delta\big(\bar w_{v\dot\alpha}\mu_v^A + w_{v\dot\alpha}\bar\mu_v^A\big) = \zeta\,, \tag{3.5}$$

for each $A = 1, 2$. The remaining variables, $\{w_{u\dot\alpha}, \mu_u^A, \bar\mu_u^A\}$, $u \neq v$, as well as $\{\mathcal{K}_{if}, \tilde{\mathcal{K}}_{fi}\}$, are all treated as gaussian fluctuations around the critical point. To this order, the instanton effective action (2.16) is

$$S = 4\pi^2 \left\{ \zeta\boldsymbol{\chi}^2 + \sum_{\substack{u=1\\(\neq v)}}^{N} \Big(\boldsymbol{\phi}_{vu}^2 \big|w_{u\dot\alpha}\big|^2 + \frac{i}{2}\phi_{vu}^\dagger \bar\mu_u^A \mu_{uA}\Big) + \frac{1}{4}\sum_{f=1}^{N_F} (m_f + \phi_v)\mathcal{K}_f \tilde{\mathcal{K}}_f \right\} + \cdots\,, \tag{3.6}$$

where $\boldsymbol{\phi}_{uv} \equiv \boldsymbol{\phi}_u - \boldsymbol{\phi}_v$. The integrals are easily done. Note that the integral over $\boldsymbol{\chi}$ yields a factor of ζ^{-1} which cancels against the factors of ζ arising from (3.4) and (3.5) so the final result is, as expected, independent of ζ. Summing over the N critical-point sets gives the centred one-instanton partition function

$$\widehat{\mathcal{Z}}_{1,N}^{(\mathcal{N}=2,N_F)} = \sum_{v=1}^{N} \left\{ \prod_{\substack{u=1\\(\neq v)}}^{N} \frac{1}{\phi_{vu}^2} \prod_{f=1}^{N_F} (m_f + \phi_v) \right\}\,. \tag{3.7}$$

Notice that the resulting expression is holomorphic in the VEVs.

At this point, we invite the reader to compare the efficacy of the localization method compared with the explicit integral performed in ref. [5].

4. The centred two-instanton partition function

We now evaluate the centred two-instanton partition function using localization. There are two kinds of critical-point sets. In both cases

$$\boldsymbol{\chi} = -\begin{pmatrix} \boldsymbol{\phi}_{u_1} & 0 \\ 0 & \boldsymbol{\phi}_{u_2} \end{pmatrix}\,, \tag{4.1}$$

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with $u_1 < u_2$, in the first case, giving a critical point set

$$\mathfrak{F} = \mathfrak{M}_{1,1}^{(\zeta)} \times \mathfrak{M}_{1,1}^{(\zeta)}/\mathbb{R}^4 \simeq \mathbb{R}^4 \,, \tag{4.2}$$

where the final form represents the relative position of two points in $\mathbb{R}^4$. The second case $u_1 = u_2$, gives a critical-point set

$$\mathfrak{F} = \widehat{\mathfrak{M}}_{2,1}^{(\zeta)} \,. \tag{4.3}$$

We now evaluate these two contribution separately. As in the one-instanton sector we choose the non-commutativity parameters as in (3.2).

4.1 Contribution from $\mathfrak{F} = \mathfrak{M}_{1,1}^{(\zeta)} \times \mathfrak{M}_{1,1}^{(\zeta)}/\mathbb{R}^4$

In the first case (4.2), on the critical-point set (2.28), the ADHM constraints are solved with

$$w_{ui\dot{\alpha}} = \sqrt{\zeta} e^{i\theta_i} \delta_{uu_i} \delta_{\dot{\alpha}1} \,, \qquad a'_n = \begin{pmatrix} Y_n & 0 \\ 0 & -Y_n \end{pmatrix} \,. \tag{4.4}$$

The two phase angles θ_i, $i = 1, 2$, are not genuine moduli since they can be separately rotated by transformations in the U(2) auxiliary group. The variables Y_n are the genuine moduli representing the relative positions of the two single non-commutative U(1) instantons. The corresponding solution of the fermionic ADHM constraints (2.4) on the critical-point set is

$$\mu^A = \bar{\mu}^A = 0 \,, \qquad \mathcal{M}'^A_\alpha = \begin{pmatrix} \rho^A_\alpha & 0 \\ 0 & -\rho^A_\alpha \end{pmatrix} \,, \tag{4.5}$$

where ρ^A_α are the superpartners of Y_n.

We now proceed to evaluate the contribution to the centred instanton partition function from the critical-point set. Starting from its linearized form (2.15), first we expand around the critical values (4.4) and (4.5). It is convenient to use the following notation for the fluctuations

$$\delta a'_n = \begin{pmatrix} 0 & Z_n \\ Z_n^* & 0 \end{pmatrix} \,, \qquad \delta\mathcal{M}'^A_\alpha = \begin{pmatrix} 0 & \sigma^A_\alpha \\ \varepsilon^A_\alpha & 0 \end{pmatrix} \tag{4.6}$$

and to make the shift

$$\boldsymbol{\chi} \to \boldsymbol{\chi} - \begin{pmatrix} \boldsymbol{\phi}_{u_1} & 0 \\ 0 & \boldsymbol{\phi}_{u_2} \end{pmatrix} \,, \tag{4.7}$$

so that $\boldsymbol{\chi} = 0$ on the critical submanifold. We then integrate over the Lagrange multipliers D^c and $\bar{\psi}^{\dot{\alpha}}_A$ which impose the ADHM constraints (2.2) and (2.4). The diagonal components of the constraints (in i, j indices) are the ADHM constraints of the two single U(1) instantons. The off-diagonal components vanish on the critical-point set and must therefore be expanded to linear order in the fluctuations. For the bosonic variables we have

$$\sqrt{\zeta} e^{-i\theta_1} (w_{u_1 2})_2 + \sqrt{\zeta} e^{i\theta_2} (w_{u_2 1})_2^* + 4i\bar{\eta}^1_{mn} Y_m Z_n = 0 \,, \tag{4.8a}$$

$$-i\sqrt{\zeta} e^{-i\theta_1} (w_{u_1 2})_2 + i\sqrt{\zeta} e^{i\theta_2} (w_{u_2 1})_2^* + 4i\bar{\eta}^2_{mn} Y_m Z_n = 0 \,, \tag{4.8b}$$

$$\sqrt{\zeta} e^{-i\theta_1} (w_{u_1 2})_1 + \sqrt{\zeta} e^{i\theta_2} (w_{u_2 1})_1^* + 4i\bar{\eta}^3_{mn} Y_m Z_n = 0 \,, \tag{4.8c}$$

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where $\bar\eta^c_{mn} = \frac{1}{2i}\,\mathrm{tr}_2(\tau^c\bar\sigma_m\sigma_n)$ are 't Hooft's η-symbols. Similarly in the Grassmann sector

$$\sqrt{\zeta}e^{i\theta_2}\bar\mu^A_{1u_2} + 2(\rho^{\alpha A}Z_{\alpha 1} - \sigma^{\alpha A}Y_{\alpha 1}) = 0\,, \tag{4.9a}$$

$$\sqrt{\zeta}e^{-i\theta_1}\mu^A_{u_1 2} + 2(\rho^{\alpha A}Z_{\alpha 2} - \sigma^{\alpha A}Y_{\alpha 2}) = 0\,, \tag{4.9b}$$

$$\sqrt{\zeta}e^{i\theta_1}\bar\mu^A_{2u_1} + 2(\varepsilon^{\alpha A}Y_{\alpha 1} - \rho^{\alpha A}Z^*_{\alpha 1}) = 0\,, \tag{4.9c}$$

$$\sqrt{\zeta}e^{-i\theta_2}\mu^A_{u_2 1} + 2(\varepsilon^{\alpha A}Y_{\alpha 2} - \rho^{\alpha A}Z^*_{\alpha 2}) = 0\,, \tag{4.9d}$$

where $Y_{\alpha\dot\alpha} = Y_n\sigma_{\alpha\dot\alpha n}$, *etc.* These equations correspond to a set of linear relations between the fluctuations. It is convenient to define

$$(w_{u_1 2})_1 = e^{i\theta_1}(\xi+\lambda)\,, \qquad (w_{u_2 1})^*_1 = e^{-i\theta_2}(-\xi+\lambda)\,, \tag{4.10}$$

so that the fluctuations ξ drops out from (4.8c). We can use (4.8a)–(4.8c) to solve for $(w_{u_1 2})_2$, $(w_{u_2 1})_2$ and λ, and (4.9a)–(4.9d) to solve for $\mu^A_{1u_2}$, $\mu^A_{2u_1}$, $\bar\mu^A_{u_1 2}$ and $\bar\mu^A_{u_2 1}$. We then use the U(2) symmetry to fix (i) the fluctuation Z_n to be orthogonal to Y_n, $Z_nY_n = 0$; and (ii) $\theta_i = 0$. The jacobian for the first part of this gauge fixing is

$$\frac{1}{\mathrm{Vol}\,\mathrm{U}(2)}\int d^{12}a' \to \frac{16}{\pi^2}\int d^4Y\, d^3Z\, d^3Z^*\, Y^2\,. \tag{4.11}$$

Now we turn to expanding the instanton effective action (2.16). First the bosonic pieces. To gaussian order around the critical point

$$S_{\mathrm{b}} = S^{(1)}_{\mathrm{b}} + S^{(2)}_{\mathrm{b}} + \cdots\,, \tag{4.12}$$

where

$$\frac{1}{4\pi^2}S^{(1)}_{\mathrm{b}} = \zeta\big(\boldsymbol{\chi}^2_{11}+\boldsymbol{\chi}^2_{22}\big)+8Y^2|\boldsymbol{\chi}_{12}|^2+2\Big|\boldsymbol{\phi}_{u_1u_2}\xi+\sqrt{\zeta}\,\boldsymbol{\chi}_{12}\Big|^2+2\boldsymbol{\phi}^2_{u_1u_2}\big(1+4\zeta^{-1}Y^2\big)|Z|^2 \tag{4.13}$$

and

$$\frac{1}{4\pi^2}S^{(2)}_{\mathrm{b}} = \sum_{i=1}^{2}\sum_{\substack{u=1\\(\neq u_1,u_2)}}^{N}\boldsymbol{\phi}^2_{uu_i}\big|w_{ui\dot\alpha}\big|^2\,. \tag{4.14}$$

In order to simplify the integration over the fluctuations, it is convenient to shift

$$\xi \to \xi - \sqrt{\zeta}\,\frac{\boldsymbol{\phi}_{u_1u_2}\cdot\boldsymbol{\chi}_{12}}{\boldsymbol{\phi}^2_{u_1u_2}} \tag{4.15}$$

and define the orthogonal decomposition

$$\boldsymbol{\chi} = \boldsymbol{\chi}^{\parallel} + \boldsymbol{\chi}^{\perp}\,, \qquad \boldsymbol{\chi}^{\perp}\cdot\boldsymbol{\phi}_{u_1u_2} = 0\,. \tag{4.16}$$

After having done this (4.13) becomes

$$\begin{aligned}\frac{1}{4\pi^2}S^{(1)}_{\mathrm{b}} =&\zeta\big(\boldsymbol{\chi}^2_{11}+\boldsymbol{\chi}^2_{22}\big)+2\zeta\big(1+4\zeta^{-1}Y^2\big)|\boldsymbol{\chi}^{\perp}_{12}|^2+8Y^2\big|\boldsymbol{\chi}^{\parallel}_{12}\big|^2+\\ &+2\boldsymbol{\phi}^2_{u_1u_2}\Big(|\xi|^2+\big(1+4\zeta^{-1}Y^2\big)|Z|^2\Big)\end{aligned} \tag{4.17}$$

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To gaussian order the Grassmann parts of the instanton effective action (2.16) are

$$S_{\rm f}=S_{\rm f}^{(1)}+S_{\rm f}^{(2)}+\cdots\,, \tag{4.18}$$

where

$$\begin{aligned}\frac{1}{4\pi^2}S_{\rm f}^{(1)} &= -\frac{i}{2}\phi^\dagger_{u_1u_2}\big(1+4\zeta^{-1}Y^2\big)\sigma^{\alpha A}\varepsilon_{\alpha A}+i\rho^{\alpha A}\big(2\zeta^{-1}\phi^\dagger_{u_1u_2}Z_{\alpha\dot\alpha}\bar Y^{\dot\alpha\beta}+\chi^\dagger_{12}\delta_\alpha{}^\beta\big)\varepsilon_{\beta A}+\\ &\quad+i\sigma^{\alpha A}\big(2\zeta^{-1}\phi^\dagger_{u_1u_2}Y_{\alpha\dot\alpha}\bar Z^{*\dot\alpha\beta}-\chi^\dagger_{21}\delta_\alpha{}^\beta\big)\rho_{\beta A}-2i\zeta^{-1}\phi^\dagger_{u_1u_2}\rho^{\alpha A}Z_{\alpha\dot\alpha}\bar Z^{*\dot\alpha\beta}\rho_{\beta A}\end{aligned} \tag{4.19}$$

and

$$\frac{1}{4\pi^2}S_{\rm f}^{(2)}=\frac{i}{2}\sum_{i=1}^{2}\sum_{\substack{u=1\\(\neq u_1,u_2)}}^{N}\phi^\dagger_{uu_i}\bar\mu^A_{iu}\mu_{uiA}+\frac14\sum_{i=1}^{2}\sum_{f=1}^{N_F}(m_f+\phi_{u_i})\mathcal{K}_{if}\tilde{\mathcal{K}}_{fi}\,. \tag{4.20}$$

By shifting the fluctuations σ^A and ε^A by the appropriate amounts of ρ^A, we can complete the square yielding

$$\begin{aligned}\frac{1}{4\pi^2}S_{\rm f}^{(1)} &= -\frac{i}{2}\phi^\dagger_{u_1u_2}\big(1+4\zeta^{-1}Y^2\big)\sigma^{\alpha A}\varepsilon_{\alpha A}-\\ &\quad-2i\zeta^{-1}(1+4\zeta^{-1}Y^2)^{-1}\rho^{\alpha A}\Big(\phi^\dagger_{u_1u_2}Z_{\alpha\dot\alpha}\bar Z^{*\dot\alpha\beta}+2\chi^\dagger_{12}Y_{\alpha\dot\alpha}\bar Z^{*\dot\alpha\beta}+2\chi^\dagger_{21}Z_{\alpha\dot\alpha}\bar Y^{\dot\alpha\beta}\Big)\rho_{\beta A}.\end{aligned} \tag{4.21}$$

Before we proceed, let us remind ourselves that only the variables Y_n and ρ^A_α are facets of the critical-point set, the remaining variables are all fluctuations. The contribution to the centred instanton partition function from the critical-point set is then proportional to

$$\begin{aligned}&\int d^4Y\,d\xi\,d\xi^*\,d^3Z\,d^3Z^*\,d^8\boldsymbol{\chi}\,d^4\rho\,d^4\sigma\,d^4\varepsilon\,\times\\ &\times\prod_{i=1}^{2}\Bigg\{\prod_{\substack{u=1\\(\neq u_1,u_2)}}^{N}d^2w_{ui}d^2\bar w_{iu}\,d^2\mu_{ui}d^2\bar\mu_{iu}\prod_{f=1}^{N_F}d\mathcal{K}_{if}d\tilde{\mathcal{K}}_{fi}\Bigg\}Y^2\exp\big(-S_{\rm b}^{(1)}-S_{\rm b}^{(2)}-S_{\rm f}^{(1)}-S_{\rm f}^{(2)}\big).\end{aligned} \tag{4.22}$$

The integrals over the Grassmann variables $\{\sigma^A_\alpha,\varepsilon^A_\alpha,\rho^A_\alpha\}$ are saturated by pulling down terms from $S_{\rm f}^{(1)}$ yielding the factors

$$(\phi^\dagger_{u_1u_2})^4\zeta^{-2}(1+4\zeta^{-1}Y^2)^2\Big(4Y^2(\chi^\dagger_{21}Z-\chi^\dagger_{12}Z^*)^2+(\phi^\dagger_{12})^2(Z^2Z^{*2}-(Z\cdot Z^*)^2)\Big)\,. \tag{4.23}$$

The integrals over the remaining Grassmann variables $\{\mu^A_{ui},\bar\mu^A_{iu},\mathcal{K}_{if},\tilde{\mathcal{K}}_{fi}\}$, $u\neq u_1,u_2$, are saturated by pulling down terms from $S_{\rm f}^{(2)}$ giving rise to

$$\prod_{i=1}^{2}\prod_{\substack{u=1\\(\neq u_1,u_2)}}^{N}(\phi^\dagger_{uu_i})^2\prod_{f=1}^{N_F}(m_f+\phi_{u_1})(m_f+\phi_{u_2})\,. \tag{4.24}$$

The $\{Z,\xi,\boldsymbol{\chi}\}$ integrals are

$$\begin{aligned}&\int d\xi\,d\xi^*\,d^3Z\,d^3Z^*\,d^8\boldsymbol{\chi}\,\Big(4Y^2(\chi^\dagger_{21}Z-\chi^\dagger_{12}Z^*)^2+(\phi^\dagger_{u_1u_2})^2(Z^2Z^{*2}-(Z\cdot Z^*)^2)\Big)\,e^{-S_b^{(1)}}=\\ &=2^{-21}3\pi^{-8}\frac{(\phi^\dagger_{u_1u_2})^2}{\zeta^3\boldsymbol{\phi}^{12}_{u_1u_2}Y^2(1+4\zeta^{-1}Y^2)^6}\end{aligned} \tag{4.25}$$

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while those over $w_{ui\dot{\alpha}}$, $u \neq u_1, u_2$, give a factor

$$\prod_{i=1}^{2} \prod_{\substack{u=1 \\ (\neq u_1,u_2)}}^{N} \frac{1}{\boldsymbol{\phi}_{uu_i}^4} \,. \tag{4.26}$$

Finally all that remains is to integrate over the relative position of the instantons:

$$\int d^4Y \, \frac{1}{\zeta^2(1+4\zeta^{-1}Y^2)^4} = \frac{\pi^2}{96} \,. \tag{4.27}$$

Putting all the pieces together with the correct numerical factors gives the final contribution of the critical-point set to the centred instanton partition function

$$\frac{2}{\phi_{u_1u_2}^6} \prod_{i=1}^{2} \prod_{\substack{u=1 \\ (\neq u_1,u_2)}}^{N} \frac{1}{\phi_{uu_i}^2} \prod_{f=1}^{N_F} (m_f + \phi_{u_1})(m_f + \phi_{u_2}) \,. \tag{4.28}$$

Notice that the result is holomorphic in the VEVs as required. Summing over the $\frac{1}{2}N(N-1)$ critical-point sets of this type gives the following contribution

$$\sum_{\substack{u,v=1 \\ (u\neq v)}}^{N} \frac{S_u(\phi_u)S_v(\phi_v)}{\phi_{uv}^2} \,, \tag{4.29}$$

where we have written the answer in terms of the functions

$$S_u(x) \equiv \prod_{\substack{v=1 \\ (\neq u)}}^{N} \frac{1}{(x-\phi_v)^2} \prod_{f=1}^{N_F} (m_f + x) \tag{4.30}$$

defined in [34].

4.2 Contribution from $\mathfrak{F} = \widehat{\mathfrak{M}}_{2,1}^{(\zeta)}$

There are N critical-point sets of this type for which $u_1 = u_2 \equiv v \in \{1, \ldots, N\}$. On the critical submanifold $\{w_{vi\dot{\alpha}}, a'_n\}$ and $\{\mu^A_{vi}, \bar{\mu}^A_{iv}, \mathcal{M}'^A_{\alpha}\}$ satisfy the ADHM constraints, (2.2) and (2.4), respectively, of two instantons in a non-commutative U(1) theory. The remaining variables all vanish and are treated as fluctuations around the critical-point set.

As previously, it is convenient to shift the auxiliary variable $\boldsymbol{\chi}$ by its critical-point value:

$$\boldsymbol{\chi} \to \boldsymbol{\chi} - \boldsymbol{\phi}_v 1_{[2]\times[2]} \,. \tag{4.31}$$

We now expand in the fluctuations $\{w_{ui\dot{\alpha}}, \mu^A_{ui}, \bar{\mu}^A_{iu}\}$, for $u \neq v$, as well as $\{\mathcal{K}_{if}, \tilde{\mathcal{K}}_{fi}\}$. Since all the components of the ADHM constraints are non-trivial at leading order the fluctuations decouple from the δ-functions in (2.8) and (2.10) which impose the constraints. The fluctuation integrals only involve the integrand $\exp -S$, where S is expanded to gaussian order around the critical-point set. However, it is important, as we shall see below, to leave $\boldsymbol{\chi}$ arbitrary rather than set it to its critical-point value; namely, $\boldsymbol{\chi} = 0$, after the shift (4.31). The fluctuation integrals produce the non-trivial factor

$$\prod_{\substack{u=1\\(\neq v)}}^{N} \frac{1}{\left(\det_2(\chi + \phi_{uv}1_{[2]\times[2]})\right)^2} \prod_{f=1}^{N_F} \det_2\left((m_f+\phi_v)1_{[2]\times[2]}-\chi\right) = S_v(\phi_v-\lambda_1)S_v(\phi_v-\lambda_2)\,. \quad (4.32)$$

Here, λ_i, $i = 1,2$, are the eigenvalues of the 2×2 matrix χ and $S_u(x)$ was defined in (4.30).

The remaining integrals involve the supersymmetric volume integral on $\widehat{\mathfrak{M}}^{(\zeta)}_{2,1}$, into which we insert the integrand (4.32) which depends non-trivially on χ. Now by itself $\int_{\widehat{\mathfrak{M}}^{(\zeta)}_{2,1}} \boldsymbol{\omega}^{(\mathcal{N}=2)} = 0$. This is clear from the linearized form (2.15): integrals over the Grassmann collective coordinates pull down two elements of the matrix $\chi^\dagger$ from the action and since there are no compensating factors of χ the resulting integrals over the phases of the elements of χ will integrate to zero. This is why we left χ arbitrary in (4.32) since after expanding in powers of the eigenvalues λ_i it is potentially the quadratic terms that will give a non-zero result when inserted into $\int_{\widehat{\mathfrak{M}}^{(\zeta)}_{2,1}} \boldsymbol{\omega}^{(\mathcal{N}=2)}$. To quadratic order (4.32) is

$$\frac{1}{2}S_v(\phi_v)\frac{\partial^2 S_v(\phi_v)}{\partial\phi_v^2}(\lambda_1^2+\lambda_2^2) + \frac{\partial S_v(\phi_v)}{\partial\phi_v}\frac{\partial S_v(\phi_v)}{\partial\phi_v}\lambda_1\lambda_2\,. \quad (4.33)$$

So the contribution from this critical-point set is of the form

$$\mathcal{I}_1 S_v(\phi_v)\frac{\partial^2 S_v(\phi_v)}{\partial\phi_v^2} + \mathcal{I}_2\frac{\partial S_v(\phi_v)}{\partial\phi_v}\frac{\partial S_v(\phi_v)}{\partial\phi_v}\,, \quad (4.34)$$

where the VEV-independent constants $\mathcal{I}_{1,2}$ are given by the following integrals (in the linearized form (2.15))

$$\mathcal{I}_1 = \frac{1}{2}\int_{\widehat{\mathfrak{M}}^{(\zeta)}_{2,1}} \boldsymbol{\omega}^{(\mathcal{N}=2)}\,(\lambda_1^2+\lambda_2^2) \equiv \int_{\widehat{\mathfrak{M}}^{(\zeta)}_{2,1}} \boldsymbol{\omega}^{(\mathcal{N}=2)}\left(\frac{1}{2}(\mathrm{tr}_2\,\chi)^2 - \det_2\chi\right), \quad (4.35)$$

$$\mathcal{I}_2 = \int_{\widehat{\mathfrak{M}}^{(\zeta)}_{2,1}} \boldsymbol{\omega}^{(\mathcal{N}=2)}\,\lambda_1\lambda_2 \equiv \int_{\widehat{\mathfrak{M}}^{(\zeta)}_{2,1}} \boldsymbol{\omega}^{(\mathcal{N}=2)}\det_2\chi\,. \quad (4.36)$$

We remark that (4.34) is holomorphic in the VEVs as required.

The moduli space $\widehat{\mathfrak{M}}^{(\zeta)}_{2,1}$ is the Eguchi-Hanson manifold [18], a well-known four-dimensional hyper-Kähler space. So after all the Grassmann variables and $\boldsymbol{\chi}$ have been integrated out, we can write $\mathcal{I}_{1,2}$ as integrals over the Eguchi-Hanson space of a suitable integrand. The Appendix is devoted to proving

$$\mathcal{I}_1 = \frac{1}{4}\,, \qquad \mathcal{I}_2 = 0\,. \quad (4.37)$$

Hence the final result for the contributions from the N critical points of this type to the centred instanton partition function is

$$\frac{1}{4}\sum_{u=1}^{N} S_u(\phi_u)\frac{\partial^2 S_u(\phi_u)}{\partial\phi_u^2}\,. \quad (4.38)$$

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Finally, summing (4.38) and (4.29) we have the centred two-instanton partition function

$$\widehat{\mathcal{Z}}_{2,N}^{(\mathcal{N}=2,N_F)} = \sum_{\substack{u,v=1\\(u\neq v)}}^{N} \frac{S_u(\phi_u)S_v(\phi_v)}{\phi_{uv}^2} + \frac{1}{4}\sum_{u=1}^{N} S_u(\phi_u)\frac{\partial^2 S_u(\phi_u)}{\partial\phi_u^2} . \tag{4.39}$$

5. The one- and two-instanton contributions to the prepotential

Now that we have calculated the centred one- and two-instanton partition functions using localization we can proceed to write down the one- and two-instanton contributions to the prepotential using (1.2). Writing the expression in terms of the quantity $S_u(x)$ defined in (4.30), we have

$$\mathcal{F}_1^{\rm nc} = \sum_{u=1}^{N} S_u(\phi_u)\,, \tag{5.1a}$$

$$\mathcal{F}_2^{\rm nc} = \sum_{\substack{u,v=1\\(u\neq v)}}^{N} \frac{S_u(\phi_u)S_v(\phi_v)}{\phi_{uv}^2} + \frac{1}{4}\sum_{u=1}^{N} S_u(\phi_u)\frac{\partial^2 S_u(\phi_u)}{\partial\phi_u^2}\,, \tag{5.1b}$$

where the label reminds us that the result is really calculated in the non-commutative theory.

The one-instanton contribution (5.1a) should be compared with the brute-force calculation of the one-instanton contribution reported in ref. [5][16]

$$\mathcal{F}_1 = \prod_{\substack{u,v=1\\(u\neq v)}}^{N} \frac{1}{(\phi_v-\phi_u)^2}\prod_{f=1}^{N_F}(m_f+\phi_v)\Big\}+S_1^{(N_F)}\,. \tag{5.2}$$

The extra contribution compared with (5.1a) is

$$\begin{aligned} S_1^{(N_F<2N-2)} &= 0\,,\\ S_1^{(2N-2)} &= -2^{2(1-N)}\begin{pmatrix}2N-3\\N-1\end{pmatrix}\,,\\ S_1^{(2N-1)} &= -2^{2(1-N)}\begin{pmatrix}2N-3\\N-1\end{pmatrix}\sum_{f=1}^{N_F} m_f\,,\\ S_1^{(2N)} &= -2^{2(1-N)}\begin{pmatrix}2N-3\\N-1\end{pmatrix}\sum_{\substack{f,f'=1\\(f<f')}}^{N_F} m_f m_{f'} - 2^{-2N}\begin{pmatrix}2N\\N-1\end{pmatrix}\sum_{u=1}^{N}\phi_u^2\,. \end{aligned} \tag{5.3}$$

There is an interesting interpretation of this extra contribution to $\mathcal{F}_1$, denoted $\mathcal{F}_\partial$ in ref. [5]. The brute-force calculation in ref. [5] involved the commutative theory and so an integral over $\widehat{\mathfrak{M}}_{1,N}$. For $k=1$, and $\zeta^c=0$, the ADHM constraints (2.2) are solved with

$$w = \rho\,\Omega\begin{pmatrix}1_{[2]\times[2]}\\0_{[N-2]\times[2]}\end{pmatrix}\,, \tag{5.4}$$

[16] In order to compare with this reference our VEVs should be multiplied by $-\sqrt{2}$.

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where $\Omega \in \mathrm{SU}(N)$ and we think of w as a $N \times 2$ matrix with elements $w_{u\dot{\alpha}}$. The parameter ρ is identified with the instanton scale size. Taking into account the stabilizer of the group action as well as the quotient by U(1) transformations (2.1), the moduli space has the form of a cone:

$$\widehat{\mathfrak{M}}_{1,N} \simeq \mathbb{R}^+ \times \frac{\mathrm{SU}(N)}{\mathrm{S}\big(\mathrm{U}(N-2) \times \mathrm{U}(1)\big)}\,, \tag{5.5}$$

where ρ is the coordinate along the cone. This space has conical a singularity at the apex of the cone $\rho = 0$ where the instanton has shrunk to zero size. It is this singularity with is smoothly resolved in the non-commutative theory. Interestingly the extra contribution denoted $\mathcal{F}_\partial$ in ref. [5] actually arises from the singularity itself. To see this, one has to follow in detail the analysis of ref. [5]. The strategy followed is to perform the integral in the linearized form (2.15) and leave the χ (denoted z in ref. [5]) integral until last. The final χ integrals can then be performed using Gauss' Theorem. There are pole contributions which give rise to the first term in (5.2) and a boundary contribution from the circle at infinity in χ-space. It is this latter contribution which gives rise to $\mathcal{F}_\partial$. However, this region of χ-space corresponds to the singularity of $\widehat{\mathfrak{M}}_{1,N}$ as can be seen by the following argument. Consider again the linearized integral form for the partition function (2.15). Suppose we integrate out the variable χ by solving its equation-of-motion. This gives $|\chi| \sim \rho^{-1}$ so that the large circle in $\boldsymbol{\chi}$-space corresponds to $\rho = 0$. In the non-commutative theory the singularity is resolved and the contribution $S_1^{(N_F)}$ disappears. In fact using the formalism of [5] with the addition of the non-commutativity parameters ζ^c one can show (as was done for the case $N_F = 0$ in [24]) that the contribution from the circle at infinity in $\boldsymbol{\chi}$-space vanishes. So we have a nice intuitive picture of the difference between the calculations in the commutative and non-commutative theories: the latter misses a contribution from the singularity corresponding to point-like instantons.

Now we compare our results to the Seiberg-Witten curves proposed in refs. [35, 36], for $N_F = 0$, and refs. [37, 38, 39], for $1 \leq N_F < 2N$ (restricted to $N \leq 3$ for the last reference). The finite theory with $N_F = 2N$ will be considered separately. Extracting even the one-instanton coefficient of the prepotential is a lengthy calculation undertaken in refs. [34, 40]. It is interesting that the Seiberg-Witten theory predictions precisely match the expression (5.1a) of the non-commutative theory. In other words, the predictions do not re-produce the additional contribution $S_1^{(N_F)}$ which, as we have argued, arises from the singularity of the moduli space. However, for $N_F < 2N$ this contribution is only a constant (VEV-independent) contribution to the prepotential which is entirely unphysical since the low-evergy effective action only depends on derivatives of the prepotential. Now we turn to the two-instanton sector with $N_F < 2N$. If the picture at the one-instanton level is matched at the two-instanton level then we should expect (5.1b) to match the prediction from Seiberg-Witten theory. The two-instanton contribution was extracted from the curve via a lengthy calculation in ref. [34]. It is astonishing to find exact agreement with our calculation of $\mathcal{F}_2$ using localization (5.1b). Hence, the main conclusion we can draw is that this is strong evidence in favour of the hypothesis made in ref. [24] that the Seiberg-Witten predictions for the prepotential in the commutative and non-commutative theories are identical.

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The case with $N_F = 2N$ is rather special since the underlying theory in this case is finite. There are two issues here; namely the relation of the, on the one hand, commutative, and on the other, non-commutative, instanton calculations, to the predictions from the Seiberg-Witten curves. The relation of the commutative instanton calculations to the prediction of the Seiberg-Witten curve has already been extensively studied [6, 7, 41, 42]. In particular, we believe that a general picture has been established in the latter reference. Of particular importance is the notion of re-parameterizations of the couplings, masses and VEVs. Since it is known that conventional commutative instanton calculations are consistent with the curves, we find it more convenient to simply compare the commutative and non-commutative instanton calculations. We find that for identical infra-red physics the the commutative and non-commutative theories should be formulated in terms of two different coupling constants, τ and $\tau_{\rm nc}$ with

$$\tau_{\rm nc} = \tau + \sum_{k=1}^{\infty} c_k e^{2\pi i k\tau} \,, \tag{5.6}$$

where the c_k are constants. For the commutative theory with $N = 2N_F$ the prepotential has the semi-classical expansion

$$\mathcal{F}\Big|_{N_F=2N} = \frac{1}{2}\tau\sum_{u=1}^{N}\phi_u^2 + \frac{1}{2\pi i}\sum_{k=1}^{\infty}\mathcal{F}_k e^{2\pi i k\tau} \,. \tag{5.7}$$

The expansion is identical in the non-commutative theory with τ replaced by $\tau_{\rm nc}$. By the hypothesis of [24], the prepotentials in both cases can only differ by a physically irrelevant VEV-independent piece, hence

$$\mathcal{F}_1 = \mathcal{F}_1^{\rm nc} + i\pi c_1 \sum_{u=1}^{N}\phi_u^2 \,, \tag{5.8a}$$

$$\mathcal{F}_2 = \mathcal{F}_2^{\rm nc} + 2\pi i c_1 \mathcal{F}_1^{\rm nc} + i\pi c_2 \sum_{u=1}^{N}\phi_u^2 \,, \tag{5.8b}$$

etc, modulo VEV-independent constants. Comparing our result (5.1a) with the brute-force calculation in ref. [5] gives

$$c_1 = -\frac{1}{2^{2N} i\pi}\begin{pmatrix} 2N \\ N-1 \end{pmatrix} . \tag{5.9}$$

For the case SU(2) we can also compare (5.1b) with the explicit brute-force integration over the two instanton moduli space described in ref. [7]. One then finds

$$c_2\Big|_{\rm SU(2)} = \frac{1}{2^3\pi i}\Big(1 + \frac{7}{2^4 3^5} - \frac{13}{2^4}\Big) . \tag{5.10}$$

Note that the consistency of (5.8a) and (5.8b) is non-trivial because the coefficients $\mathcal{F}_{1,2}$ and $\mathcal{F}_{1,2}^{\rm nc}$ all depend non-trivially on the masses. Notice that consistency at the two-instanton level is maintained by assuming that the masses in commutative and non-commutative theories are equal.

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What about the situation for instanton number $k > 2$? It is clear that the localization technique can be extended beyond $k > 2$. The answer for $\mathcal{F}_k$ will then be written as a sum over the contributions from the different critical-point sets (2.29) and each contribution will involve an integral over the product of U(1) instanton moduli spaces. The holy grail of this quest would be to reproduce the recursion relations of refs. [43, 44, 45, 46] for the coefficients $\mathcal{F}_k$ and thereby completely prove Seiberg-Witten theory using conventional semi-classical field theory methods.

Finally we should mention that these localization techniques are also applicable to calculating the instanton coefficients of the prepotential of the $\mathcal{N} = 2$ theory that arises from a mass-deformation of the $\mathcal{N} = 4$ theory. This situation is considered in ref. [4].

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A. Integrals on the Eguchi-Hanson manifold

The space $\widehat{\mathfrak{M}}^{(\zeta)}_{2,1}$ is the Eguchi-Hanson manifold as can be seen by explicitly solving the ADHM constraints. Here we follow the treatment in ref. [18]. Up to (most of) the U(2) symmetry

$$w_1 = \begin{pmatrix} \sqrt{1-b} & \sqrt{1+b} \end{pmatrix} , \qquad w_2 = 0 , \qquad a'_n = y_n \begin{pmatrix} 1 & \sqrt{2b/a} \\ 0 & -1 \end{pmatrix} , \tag{A.1}$$

where

$$a = \frac{1}{2}\zeta^{-1}y^2 , \qquad b = \frac{1}{a + \sqrt{1+a^2}} . \tag{A.2}$$

Defining coordinates

$$z_0 \equiv y_2 - iy_1 = r\cos\frac{\theta}{2}e^{i(\psi+\varphi)/2} , \qquad z_1 \equiv y_4 - iy_3 = r\sin\frac{\theta}{2}e^{i(\psi-\varphi)/2} , \tag{A.3}$$

we find

$$\int_{\widehat{\mathfrak{M}}^{(\zeta)}_{2,1}} \boldsymbol{\omega} = 2\pi^2 \int r^3 dr\, \sin\theta\, d\theta\, d\varphi\, d\psi , \tag{A.4}$$

where $0 \leq \theta \leq \pi$, $0 \leq \varphi, \psi \leq 2\pi$.

The solution of the fermionic ADHM constraints can be written as

$$\begin{aligned} \mathcal{M}'^{1A} &= \begin{pmatrix} \sqrt{\frac{a}{2b}}(z_0\sigma^A + z_1\varepsilon^A) & z_0\sigma^A \\ z_1\varepsilon^A & -\sqrt{\frac{a}{2b}}(z_0\sigma^A + z_1\varepsilon^A) \end{pmatrix} , \\ \mathcal{M}'^{2A} &= \begin{pmatrix} \sqrt{\frac{a}{2b}}(z_1^*\sigma^A - z_0^*\varepsilon^A) & z_1^*\sigma^A \\ -z_0^*\varepsilon^A & -\sqrt{\frac{a}{2b}}(z_1^*\sigma^A - z_0^*\varepsilon^A) \end{pmatrix} , \end{aligned} \tag{A.5}$$

along with $\mu^A = \bar{\mu}^A = 0$. The Grassmann part of the integration measure (2.10) is

$$\pi^{-8}\frac{r^4 b^8}{(1-b^2)^4(1+b^2)^2}\int d^2\sigma\, d^2\varepsilon . \tag{A.6}$$

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The centred partition function, written using the auxiliary variables $\boldsymbol{\chi}$ (2.15), is then

$$\widehat{\mathcal{Z}}_{2,1} = 2^4 \int r^3 dr\, d^3\Omega\, d^2\sigma\, d^2\varepsilon\, d^8\boldsymbol{\chi}\, \frac{r^4 b^6}{(1-b)^4(1+b)^2(1+b^2)}\, e^{-S}\,. \tag{A.7}$$

The instanton effective action is

$$S = 4\pi^2 \chi \boldsymbol{L} \chi^\dagger - 2i\pi^2 r^2 \varepsilon^A \sigma_A \left(\chi_{11}^\dagger - \sqrt{\frac{2a}{b}} \chi_{12}^\dagger - \sqrt{\frac{2a}{b}} \chi_{21}^\dagger - \chi_{22}^\dagger \right), \tag{A.8}$$

where

$$\boldsymbol{L} = \begin{pmatrix} 1 & 0 & 0 & -b \\ 0 & 0 & 1+2a+b & \sqrt{1-b^2} \\ 0 & 1+2a+b & 0 & \sqrt{1-b^2} \\ -b & \sqrt{1-b^2} & \sqrt{1-b^2} & 1+2b \end{pmatrix} \tag{A.9}$$

in the basis where χ is to thought of as the 4-vector $\chi = (\chi_{11}, \chi_{12}, \chi_{21}, \chi_{22})$.

Integrating over the Grassmann variables gives

$$\begin{aligned} \widehat{\mathcal{Z}}_{2,1} &= 2^7 \pi^4 \int r^3 dr\, \sin\theta\, d\theta\, d\varphi\, d\psi\, d^8\boldsymbol{\chi} \frac{r^8 b^6}{(1-b)^4(1+b)^2(1+b^2)} \times \\ &\quad \times \left(\chi_{11}^\dagger - \sqrt{\frac{2a}{b}} \chi_{12}^\dagger - \sqrt{\frac{2a}{b}} \chi_{21}^\dagger - \chi_{22}^\dagger \right)^2 \exp(-4\pi^2 \chi \boldsymbol{L} \chi^\dagger)\,. \end{aligned} \tag{A.10}$$

The integral over the phases of χ clearly vanish and so $\widehat{\mathcal{Z}}_{2,1} = 0$.

Now we consider the two integrals defined in (4.36). Inserting the general factor $F(\chi_{ij})$, we can perform the integrals over $\boldsymbol{\chi}$ and the angles to arrive at

$$\begin{aligned} &\int r^3 dr\, \frac{r^8 b^8}{(1-b^2)^4(1+b^2)^2} \left(\frac{\partial}{\partial J_{11}} - \sqrt{\frac{2a}{b}} \frac{\partial}{\partial J_{12}} - \sqrt{\frac{2a}{b}} \frac{\partial}{\partial J_{21}} - \frac{\partial}{\partial J_{22}} \right)^2 \times \\ &\quad \times F\left(\frac{\partial}{\partial J_{ij}^\dagger} \right) e^{J \boldsymbol{L}^{-1} J^\dagger} \Big|_{J=0}\,. \end{aligned} \tag{A.11}$$

Using this general formula one finds

$$\mathcal{I}_1 = 4 \int_0^\infty dr \frac{r^{11} b^8}{(1-b^4)^4} = \frac{1}{4}\,, \qquad \mathcal{I}_2 = 0\,, \tag{A.12}$$

where b is a function of r given in (A.2).

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